Ekman Flow:

Assume that the Coriolis force and friction force are the dominant terms in the horizontal component equations of motion. Also assume that f and A_z are constants. With the first two assumptions, here are the relevant horizontal equations of motion:

$$0 = f \cdot v + A_z \frac{d^2 u}{dz^2}$$

$$0 = -f \cdot u + A_z \frac{d^2 v}{dz^2}$$

Employing a trick similar to one used to solve for the Inertial Flow, we multiply the bottom equation by i, add the two equations, obtain the sum, multiply the left-hand term by i^2 and the right-hand term by -1.

$$0 = i^2 f(-iu + v) - A_z \frac{d^2(u+iv)}{dz^2} \qquad \text{or} \qquad 0 = if(u+iv) - A_z \frac{d^2(u+iv)}{dz^2}$$

Rearranging:

$$0 = \frac{d^2(u+iv)}{dz^2} - \frac{if}{A_z}(u+iv)$$

Here we contrast the mathematics of Inertial and Ekman Flow:

Inertial Flow:

$$0 = \frac{d(v+iu)}{dt} - if(v+iu)$$

First order ordinary differential equation

Independent variable is "t"

Characteristic Equation: $B - if = 0$

$$v + ui = h_o \exp(ift)$$

h_o is "yet-to-be determined"

Ekman Flow:

$$0 = \frac{d^2(u+iv)}{dz^2} - \frac{if}{A_z}(u+iv)$$

Second order ordinary differential equation

Independent variable is "z"

Characteristic Equation: $B^2 - \frac{if}{A_z} = 0$

$$u + iv = C_1 \exp\left(\left(\frac{if}{A_z}\right)^{1/2} z\right) + C_2 \exp\left(-\left(\frac{if}{A_z}\right)^{1/2} z\right)$$

C_1 and C_2 are "yet-to-be determined"

There is an algebraic trick that we need to deal with:

$$\sqrt{i} = \frac{i+1}{\sqrt{2}}$$ (Easily derived: $(i+1)(i+1) = ?$)

Hence, for the "B" in the Ekman solution

$$B = \left(\frac{if}{A_z}\right)^{1/2} = \frac{i+1}{\sqrt{2}}\left(\frac{f}{A_z}\right)^{1/2} = (i+1)\left(\frac{f}{2A_z}\right)^{1/2}$$

Here is result:

$$B = (i+1)\left(\frac{f}{2A_z}\right)^{1/2} \tag{6}$$

Boundary Condition #1–

@ $z=-\infty$ $\qquad u+iv=0$

i.e., no Ekman flow at large depth

Hence,

$$0=C_1\exp(B\cdot(-\infty))+C_2\exp(-B\cdot(-\infty))$$
$$0=C_2\exp(B\cdot\infty)$$

For the proceeding statement to be "true", the C_2 coefficient must equal to zero, hence
$C_2=0$

The solution to the differential equation is therefore:

$$u+iv=C_1\exp(Bz) \tag{7}$$

Note the value of the derivative, evaluated at z = 0

$$\left\{\frac{d(u+iv)}{dz}\right\}_{z=0}=BC_1\exp(0) \quad \text{or} \quad \left\{\frac{d(u+iv)}{dz}\right\}_{z=0}=BC_1 \tag{8}$$

Boundary Condition #2 –

This boundary condition is determined by the interaction of the wind with the surface water flow. The interaction is expressed in terms of the surface stress vector. The surface stress vector has the following components

$$T_x = \rho A_z \left\{ \frac{du}{dz} \right\}_{z=0}$$

$$T_y = \rho A_z \left\{ \frac{dv}{dz} \right\}_{z=0}$$

The above information can be formulated as a vector with imaginary and real components. This is done by multiplying the bottom equation by i and summing the two equations, hence

$$T_x + iT_y = \rho A_z \left\{ \frac{d(u+iv)}{dz} \right\}_{z=0} \quad \text{or} \quad \frac{T_x + iT_y}{\rho A_z} = \left\{ \frac{d(u+iv)}{dz} \right\}_{z=0} \tag{9}$$

Combining (8) and (9)

$$\frac{T_x + iT_y}{\rho A_z} = BC_1 \qquad \text{or} \qquad C_1 = \frac{T_x + iT_y}{\rho A_z B} \tag{10}$$

Combining (6) and (10) we have

$$C_1 = \frac{T_x + iT_y}{\rho A_z (i+1)\left(\dfrac{f}{2A_z}\right)^{1/2}}$$

The above is simplified with the identity, $(1+i)(1-i) = 2$

$$C_1 = \frac{(T_x + iT_y)\cdot(1-i)}{2\rho A_z \left(\dfrac{f}{2A_z}\right)^{1/2}} \tag{11}$$

Note: The parameters B (Equation 6) and C_1 (Equation 11) have both “real” and “imaginary” components

For convenience, define a constant with no imaginary component:

$$\alpha = \left(\frac{f}{2A_z} \right)^{1/2} \tag{12}$$

Combining (7), (11) and (12)

$$u + iv = \frac{T_x + T_y + i(T_y - T_x)}{2\rho A_z \alpha} exp(\alpha z) exp(i\alpha z) \tag{13}$$

As in the Inertial Flow problem, we now apply Euler's Theorem,

$$exp(i \cdot \theta) = cos(\theta) + i\, sin(\theta) \tag{14}$$

Combining (13) and (14),

$$u + iv = \frac{T_x + T_y + i(T_y - T_x)}{2\rho A_z \alpha} exp(\alpha z)\big(cos(\alpha z) + i\, sin(\alpha z)\big) \tag{15}$$

$$u(z) = \mathrm{Real}(\mathrm{Equation}\ 15) = \frac{\exp(\alpha z)}{2\rho A_z \alpha}((T_x + T_y)\cos(\alpha z) - (T_y - T_x)\sin(\alpha z))$$

$$v(z) = \mathrm{Imag.}(\mathrm{Equation}\ 15) = \frac{\exp(\alpha z)}{2\rho A_z \alpha}((T_y - T_x)\cos(\alpha z) + (T_x + T_y)\sin(\alpha z))$$

We have solved for the components of the Ekman velocity!

Note: The components are damped (with depth) and oscillatory (also with depth)

Caution: We are talking about horizontal flows and forces…the vertical component of the flow is not described by these equations.

However, we need to acknowledge that continents interact with the Ekman Flow. This is shown two pages down.

We conclude that Ekman Flow is the driver of upwelling within eastern boundary currents.

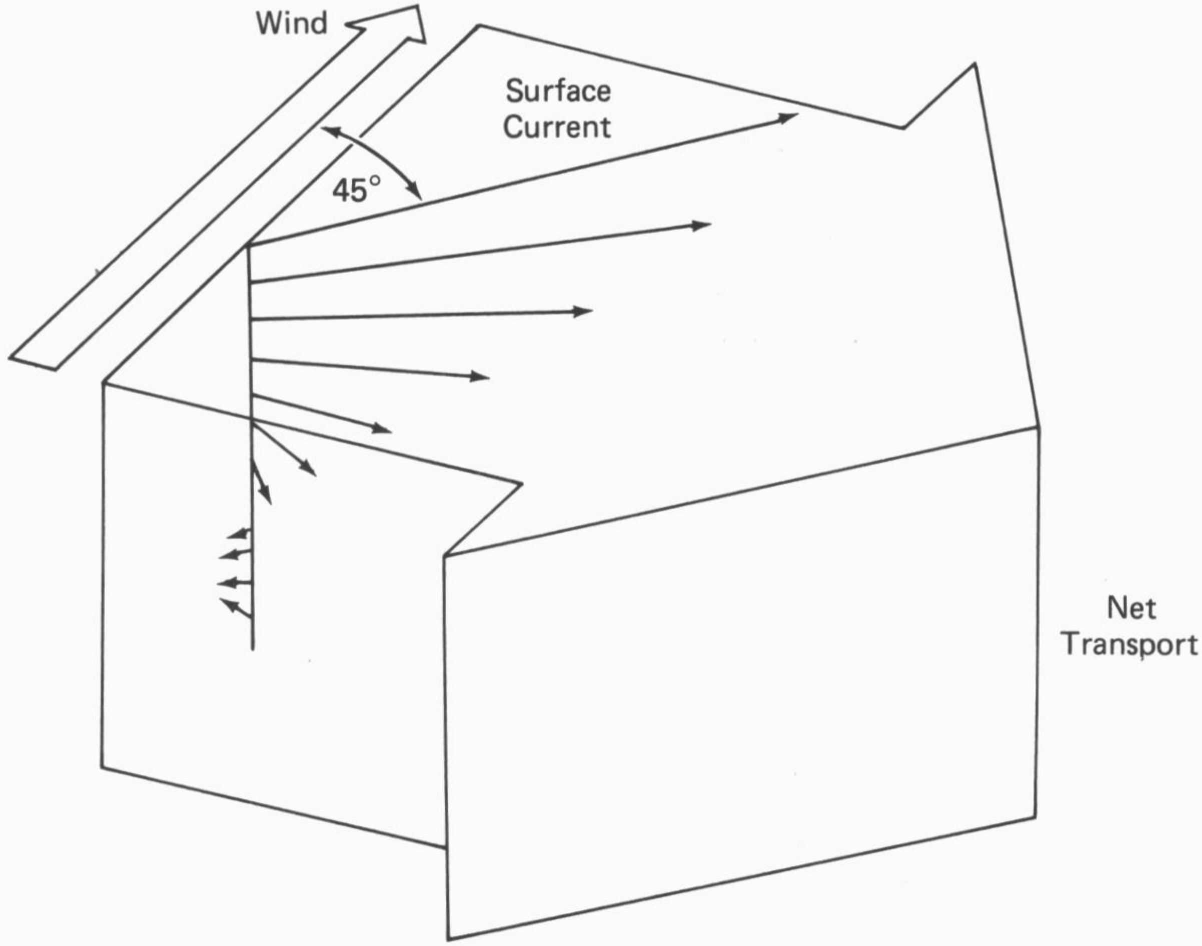


Figure 6.9 Water is set in motion by the wind. According to the Ekman relation, the effect of the Coriolis force is for each succeeding layer of water to move slightly to the right (in the Northern Hemisphere) of the layer above it. The result is the Ekman spiral as shown, with the net transport at 90° to right of the wind in the Northern Hemisphere.

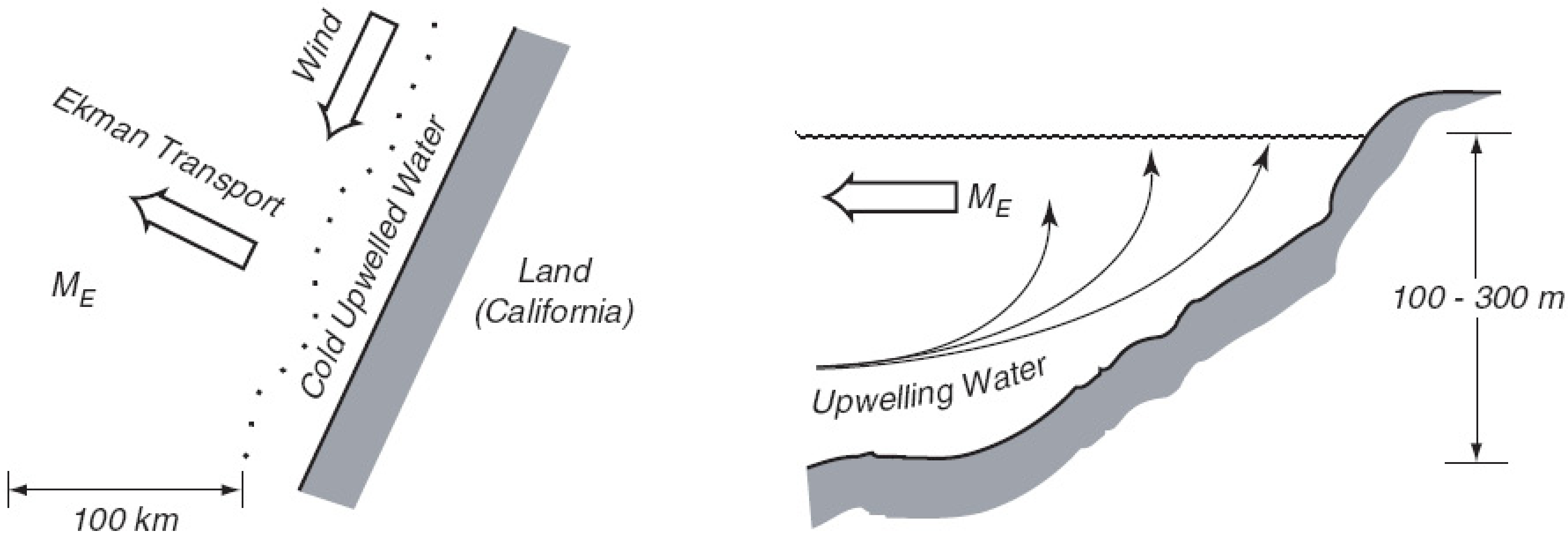


Figure 9.8 Sketch of Ekman transport along a coast leading to upwelling of cold water along the coast. **Left:** Cross section. The water transported offshore must be replaced by water upwelling from below the mixed layer. **Right:** Plan view. North winds along a west coast in the northern hemisphere cause Ekman transports away from the shore.

Summary of Ekman flow:

1. Ekman flow is directed, or "forced", by the wind. This interaction is apparent in the nature of the solution imposed by boundary condition 2

2. Stronger wind stress (i.e., larger T_x and/or larger T_y) implies stronger Ekman flow

3. In the NH, the depth-integrated horizontal mass transport due to Ekman is ~90 ^{o} *to the right* of the surface wind (Note: Figures 6.9 and 6.12 from Chapter 6 of Knauss)

4. One can think of the reciprocal of the dampening factor α as a characteristic depth (i.e., $\textit{characteristic depth} = \left(\frac{2A_z}{f}\right)^{1/2}$). Below this characteristic depth the Ekman speeds are small (damped).

5. Depending on latitude (this affects f), and turbulent intensity (proportional to A_z), the Ekman phenomenon occurs over a depth interval of tens to hundreds of meters

6. At a specified depth, the speed of the Ekman flow is smaller when the water column is either turbulent (i.e., A_z is large, implying weak stability or large wind speeds) or tropical (small f) .

7. Ekman flow also produces upwelling or downwelling at locations away from eastern boundaries. From Figure 6.12 (Knauss) we have the flowing conclusions about vertical motion within the Ekman layer

	NH	SH
Cyclonic shear	Divergence/upwelling	Convergence/downwelling
Anticyclonic shear	Convergence/downwelling	Divergence/upwelling
Cyclonic Circulation	Divergence/upwelling	Convergence/downwelling
Anticyclonic Circulation	Convergence/downwelling	Divergence/upwelling

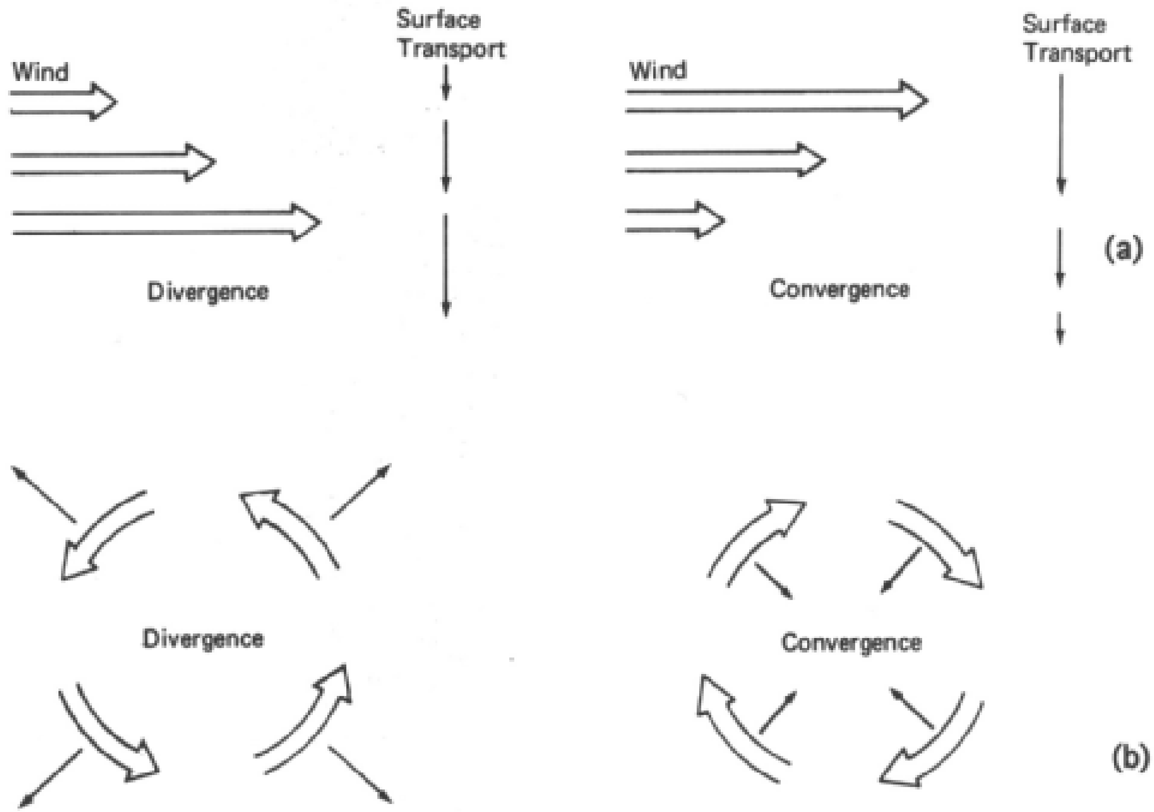
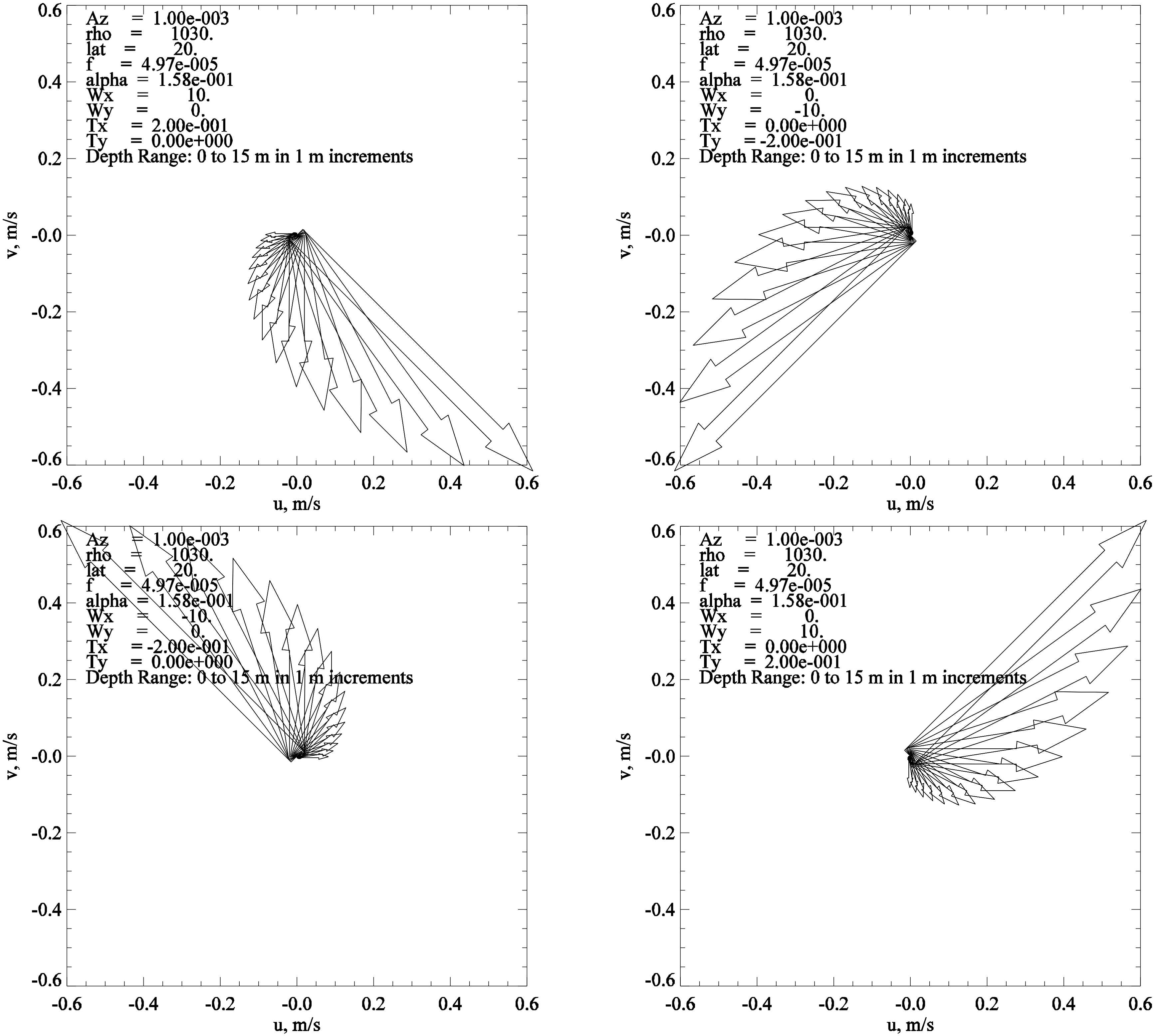


Figure 6.12 According to the Ekman relationship, horizontal wind shear can produce either convergence or divergence. (a) Cyclonic winds produce divergence in the surface waters and upwelling; and anticyclonic winds produce convergence and downwelling. (b) Similarly, atmospheric cyclones (counterclockwise in the Northern Hemisphere) cause divergence and upwelling; anticyclones cause convergence and downwelling.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rho = 1030.
lat = 20.
f = 4.97e-005
alpha = 1.58e-001
Wx = 10.
Wy = 0.
Tx = 2.00e-001
Ty = 0.00e+000
Depth Range: 0 to 15 m in 1 m increments
v, m/s
u, m/s
Az = 1.00e-003
rho = 1030.
lat = 20.
f = 4.97e-005
alpha = 1.58e-001
Wx = 0.
Wy = -10.
Tx = 0.00e+000
Ty = -2.00e-001
Depth Range: 0 to 15 m in 1 m increments
v, m/s
u, m/s
Az = 1.00e-003
rho = 1030.
lat = 20.
f = 4.97e-005
alpha = 1.58e-001
Wx = -10.
Wy = 0.
Tx = -2.00e-001
Ty = 0.00e+000
Depth Range: 0 to 15 m in 1 m increments
v, m/s
u, m/s
Az = 1.00e-003
rho = 1030.
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f = 4.97e-005
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Wx = 0.
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Tx = 0.00e+000
Ty = 2.00e-001
Depth Range: 0 to 15 m in 1 m increments
v, m/s
u, m/s

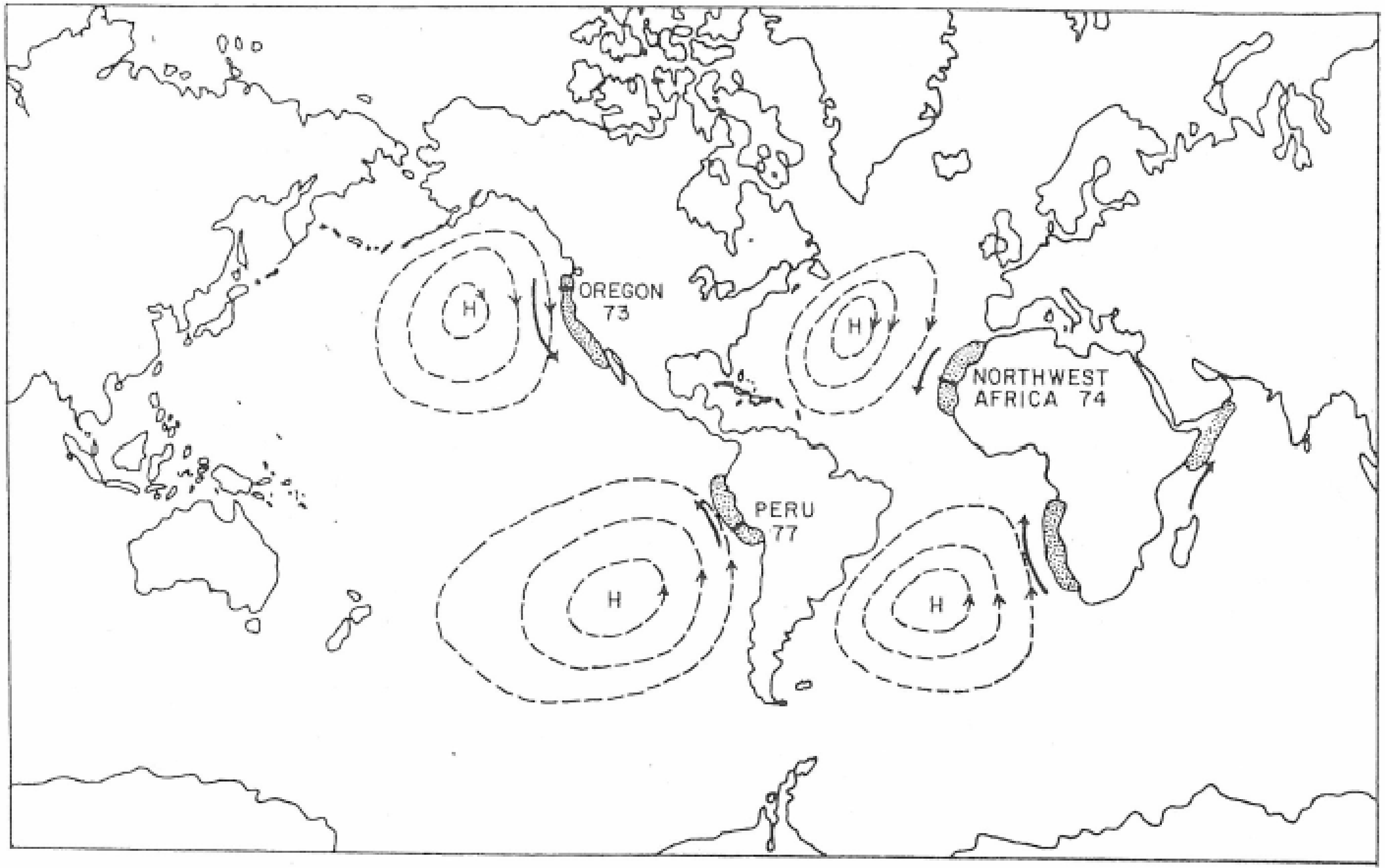

FIG. 1 Major coastal upwelling regions of the world and the sea-level atmospheric pressure systems (anticyclones) that influence them. The dashed circles represent mean idealised positions of isobars during the season of maximum upwelling in a given region. Major areas of upwelling are shown by the stippled areas; heavy lines within show the areas of study discussed in this chapter. Arrows indicate the location of the (a) California current off USA, (b) Peru current off Peru, (c) Canary current off Northwest Africa, (d) Benguela current off Southwest Africa, and (e) the Somali current in the Indian Ocean east of Africa. From Thompson (1977)