Notes ATSC 4010 Fall 2014

Calculus

In ATSC4010 we will develop several physical models. These models describe how, for example, two atmospheric properties are related. The starting point is often a physical law, expressed as a differential equation. In ATSC4010 we integrate the differential equation (physical law), subject to integration limits, and derive an integral equation describing how one atmospheric property varies with another.

The following mathematical steps are required to proceed from a physical law, expressed as a differential equation, to the integral equation (the model):

A) Separate variables
B) Identify integration limits
C) Apply integration limits
D) Integrate
E) Evaluate the antiderivatives at the integration limits
F) Using algebra, solve for one variable as a function of the other

Example from Chapter 2 (Jacob) (aka, the altimeter equation)

$$0 = \frac{-R\cdot(T_o - \Gamma\cdot z)}{P\cdot M_a}\cdot\frac{dP}{dz} - g$$

Hint, what's this called?: $0 = \frac{-1}{\rho}\cdot\frac{dP}{dz} - g$

Step A -

$$-\frac{g}{R}\cdot\frac{M_a}{(T_o - \Gamma\cdot z)}\cdot dz = \frac{1}{P}\cdot dP$$

Step B -

Lower limit: $z = 0$ and $P = P(0)$

Upper limit: $z = z$ and $P = P(z)$

Step C –

$$-\frac{g\cdot M_a}{R}\cdot\int_0^z\frac{1}{(T_o - \Gamma\cdot z)}\cdot dz = \int_{P(0)}^{P(z)}\frac{1}{P}dP$$

Step D –

$$-\frac{g\cdot M_a}{R}\cdot\left(\frac{-1}{\Gamma}\right)\cdot\left[ln(T_o - \Gamma\cdot z)\right]_0^z = \left[ln\,P\right]_{P(0)}^{P(z)}$$

Step E –

$$-\frac{g\cdot M_a}{R}\cdot\left(\frac{-1}{\Gamma}\right)\cdot\left[ln(T_o - \Gamma\cdot z) - ln(T_o)\right] = \left[ln\,P(z) - ln\,P(0)\right]$$

Step F –

$$-\frac{g\cdot M_a}{R}\cdot\left(\frac{-1}{\Gamma}\right)\cdot ln\left(\frac{T_o - \Gamma\cdot z}{T_o}\right) = ln\left(\frac{P(z)}{P(0)}\right)$$

$$\frac{g\cdot M_a}{R\cdot\Gamma}\cdot ln\left(1 - \frac{\Gamma\cdot z}{T_o}\right) = ln\left(\frac{P(z)}{P(0)}\right)$$

$$ln\left(\left(1-\frac{\Gamma \cdot z}{T_o}\right)^{\frac{g \cdot M_a}{R \cdot \Gamma}}\right) = ln\left(\frac{P(z)}{P(0)}\right)$$

$$\left(1-\frac{\Gamma \cdot z}{T_o}\right)^{\frac{g \cdot M_a}{R \cdot \Gamma}} = \frac{P(z)}{P(0)}$$

$$P(z) = P(0) \cdot \left(1-\frac{\Gamma \cdot z}{T_o}\right)^{\frac{g \cdot M_a}{R \cdot \Gamma}}$$

Also, we can write z in terms of $P(z)$

$$z = \frac{T_o}{\Gamma} \cdot \left(1-\left(\frac{P(z)}{P(0)}\right)^{\frac{R \cdot \Gamma}{g \cdot M_a}}\right)$$

Calculus in ATSC4010 – Example 1

In ATSC4010 the symbol "z" is used to describe altitude above sea level. For example, at Laramie z=2220 meters.

We know, intuitively, that temperature decreases upward. In ATSC4010, we will show that the "rate" of decrease is 10 ^oC per kilometer (0.01 ^oC per meter). Stating that result mathematically,

$$\frac{dT}{dz} = -0.01 \text{ degree per meter}$$

or

$$\frac{dT}{dz} = -0.01 \frac{^oC}{m}$$

or

$$\frac{dT}{dz} = -0.01 \; ^oC \; m^{-1}$$

Suppose the temperature and altitude at Laramie are 20 ^oC and 2220 m, respectively. Let's formulate, and plot, the function $z(T)$.

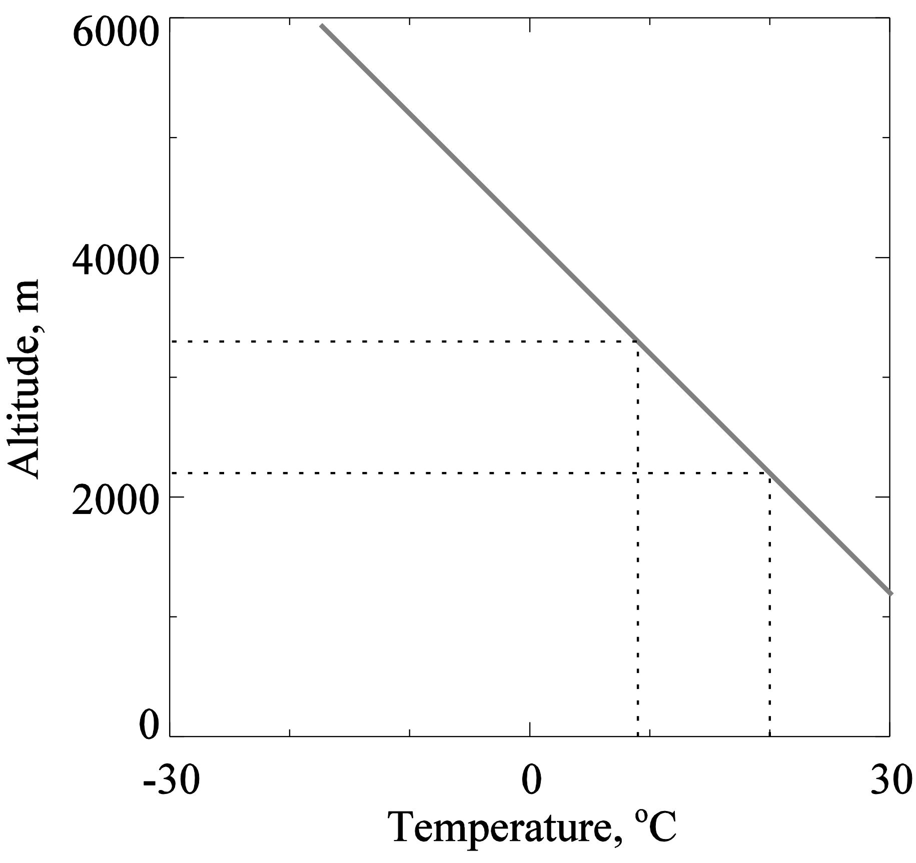
Here is the function $z(T)$:

$$z(T) = z_{LAR} + \frac{1}{b} \cdot (T - T_{LAR}) \qquad (1)$$

where z_{LAR} is 2220 m, $b = -0.01$ ^{o}C/m, $T_{LAR} = 20$ ^{o}C, and T has dimension ^{o}C.

Your first assignment is to use calculus to derive Equation 1. Start your derivation with the differential form $\frac{dT}{dz} = -0.01$, and use the integration limit $T_{LAR} = 20$ ^{o}C at $z_{LAR} = 2220$ m.

Here is a plot of altitude versus temperature:



Calculus in ATSC4010 – Example 2

In a couple of situations you will see this differential form

$$\frac{dy}{dx} = -k \cdot y \qquad (2)$$

Here y is an atmospheric property, x is a “coordinate” and k is a positive constant.

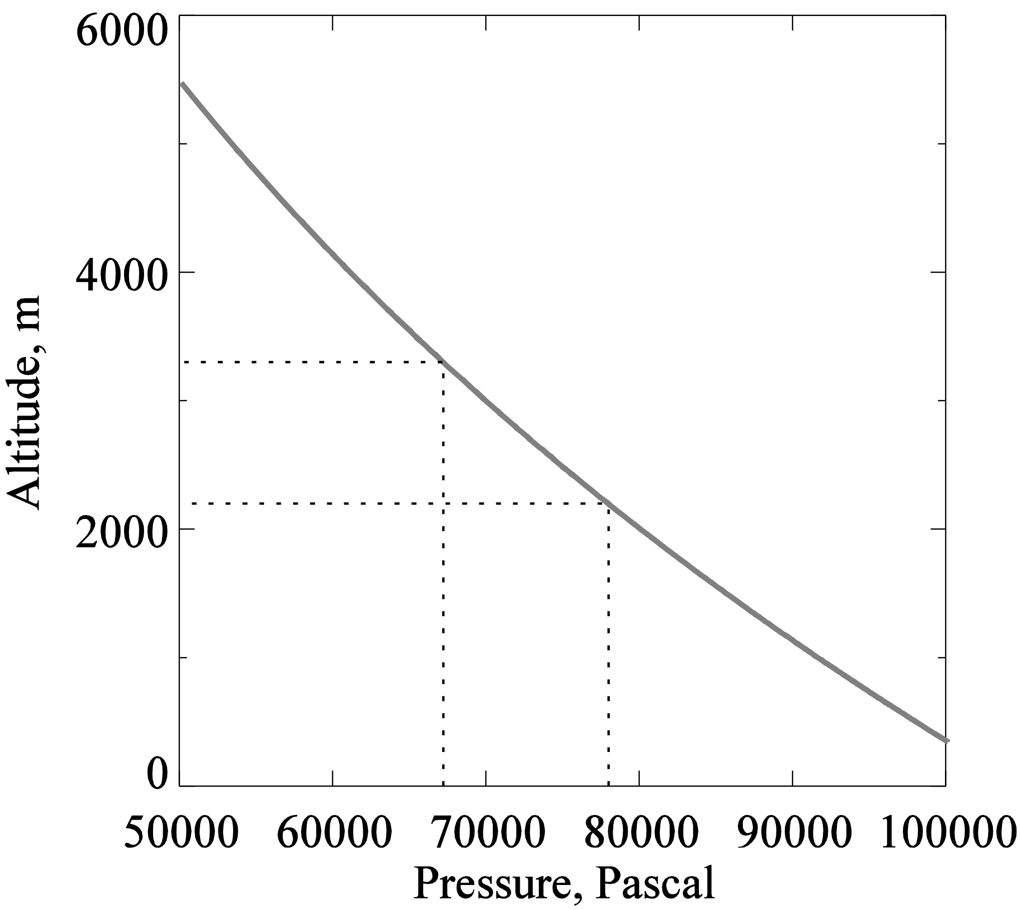
From measurements we have the integration limit x_o and y_o. Given the differential form, Equation 2, and the integration limit values, you should be able to derive the following integral form

$$x(y) = x_o - \frac{1}{k}\ln\left(\frac{y}{y_o}\right) \qquad (3)$$

Your second assignment is to use calculus to derive Equation 3 from Equation 2.

The altitude and pressure are related by a form like Equation 2 (differential) and Equation 3 (integral).

Here is a plot of altitude versus pressure:



The two previous figures demonstrate the following:

1) Temperature decreases linearly with altitude

2) Pressure decreases nonlinearly with altitude

Calculus in ATSC4010 – Example 3

$$\int_a^b \cos(\frac{t' - \alpha}{\beta}) dt' \qquad (1a)$$

$$\int_a^b \cos(\frac{t - \alpha}{\beta}) dt \qquad (1b)$$

$$= \left[\beta \cdot \sin(\frac{t-\alpha}{\beta}) \right]_a^b \qquad (2)$$

How can we check that Equations 1b and 2 are consistent?

1) Looking at Equation 2, and recalling the chain rule, we see a compound function with form $f(g(t))$ whose derivative is $\frac{df}{dg} \cdot \frac{dg}{dt}$

2) In our particular situation $f = \beta \cdot \sin(g)$ and $g(t) = \frac{t-\alpha}{\beta}$.

Hence, $\frac{df}{dg} = \beta \cdot \cos(g)$, and $\frac{dg}{dt} = \frac{1}{\beta}$. Thus, the derivative of Equation 2 is $\beta \cdot \cos(\frac{t-\alpha}{\beta}) \cdot \frac{1}{\beta} = \cos(\frac{t-\alpha}{\beta})$.

3) We see that the derivative of Equation 2 is equal to the integrand of Equation 1b. That implies that Equation 2 is the <u>antiderivative of the integrand</u> in Equation 1b. That is, via calculus, we have established that Equations 1b and 2 are equal.

Going further, we know that the notation in Equation 2 says evaluate the antiderivative at the upper limit, and subtract the antiderivative evaluated at the lower limit.   Hence,

$$\int_a^b \cos\left(\frac{t-\alpha}{\beta}\right)dt = \left(\beta \cdot \sin\left(\frac{b-\alpha}{\beta}\right) - \beta \cdot \sin\left(\frac{a-\alpha}{\beta}\right)\right)$$

Your third assignment is to evaluate the following integral:

$$\int_a^b \sin\left(\frac{t-\alpha}{\beta}\right)dt$$

Calculus in ATSC4010 – Example 4

We will see an integral expression like this:

$$\delta = \int_{a}^{x} f(x)dx$$

Further, we will seek an expression for the derivative

$$\frac{d\delta}{dx} = ?$$

Your fourth assignment is to provide an answer to the previous question.

Calculus in ATSC4010 – Example 5

Evaluate this definite integral:

$$-\int_{1}^{2} \frac{2x}{\left(9-x^2\right)^{1/3}} \cdot dx$$

Check your antiderivative by differentiating. Confirm that the antiderivative has the property that its derivative is the integrand.